Indexing Finite-State Automata Using Forward-Stable Partitions

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The Burrows-Wheeler transform

The **Burrows-Wheeler transform** (**BWT**) is a famous reversible string transformation introduced by Burrows and Wheeler in 1994 [1]. In the following years, the BWT obtained significant interest from researchers, due to the fact that it proved to be an excellent transformation to **compress** and **index** strings.

1. *Burrows M., Wheeler D.: A block-sorting lossless data compression algorithm. SRS Research Report (1994)*

Wheeler NFAs

In 2017, Gagie et al. introduced **Wheeler nondeterministic finite automata (NFAs)** [2]. Wheeler NFAs represents a **generalisation of the original BWT** to automata.

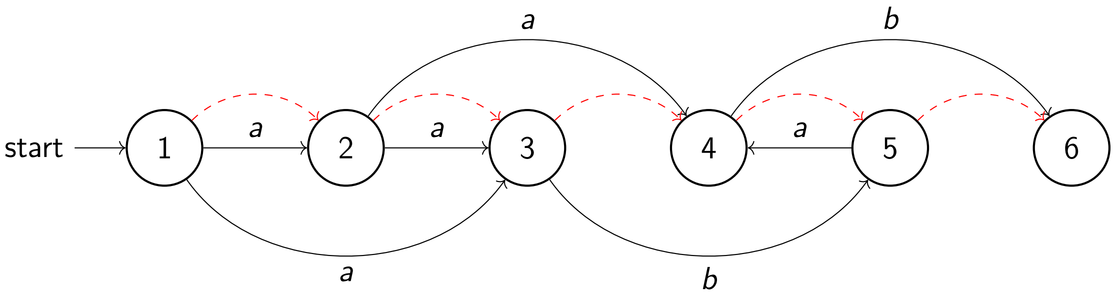
Similarly to the case of strings, **they can be compressed and indexed in almost optimal time and space**.

2. *Gagie et al.: Wheeler graphs: A framework for BWT-based data structures. Theor. Comput. Sci. (2017)*

Definition of Wheeler order

Let $\mathcal{A} = (Q, \delta, \Sigma, s)$ be an NFA. A total order $\leq$ of Q is a **Wheeler order** of $\mathcal{A}$ if for any pair $\mathbf{u} \in \delta(\mathbf{u}', \mathbf{a})$ and $\mathbf{v} \in \delta(\mathbf{v}', \mathbf{a}')$:

1. $\mathbf{a} < \mathbf{a}' \implies \mathbf{u} < \mathbf{v}$,
2. $(\mathbf{a} = \mathbf{a}') \wedge (\mathbf{u} < \mathbf{v}) \implies \mathbf{u}' \leq \mathbf{v}'$.



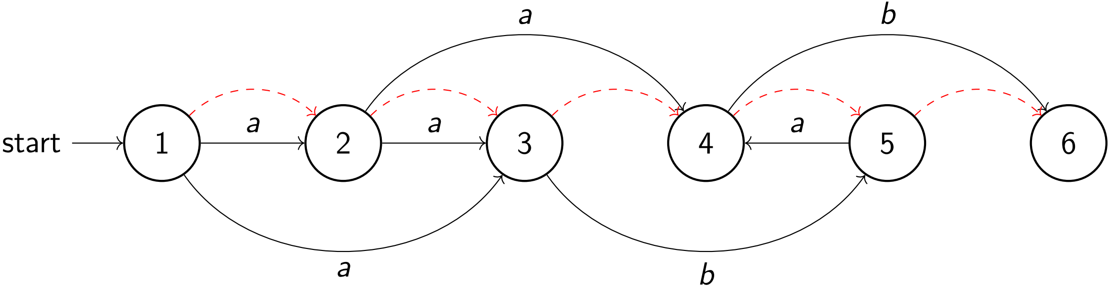
An NFA is **Wheeler** if it admits a **Wheeler order**. (NP-hard problem)

In other words...

A **Wheeler order** is a total order which sorts the states of an NFA based on the **strings reaching them**.

$$I_1 = \emptyset \qquad I_2 = \{a\} \qquad I_3 = \{a, aa\}$$

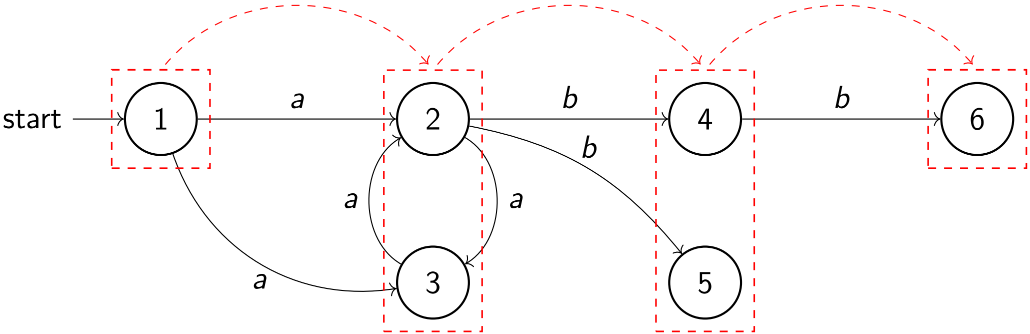
$$I_4 = \{aa, aba, aaba\} \qquad I_5 = \{ab, aab\} \qquad I_6 = \{aab, abab, aabab\}$$



We are comparing strings from **right to left**! (co-lexicographically) (NP-hard problem)

quasi-Wheeler NFAs

An NFA is **quasi-Wheeler** if it admits a **Wheeler preorder** [3]. The class of quasi-Wheeler NFAs is **strictly larger** than that of Wheeler NFAs



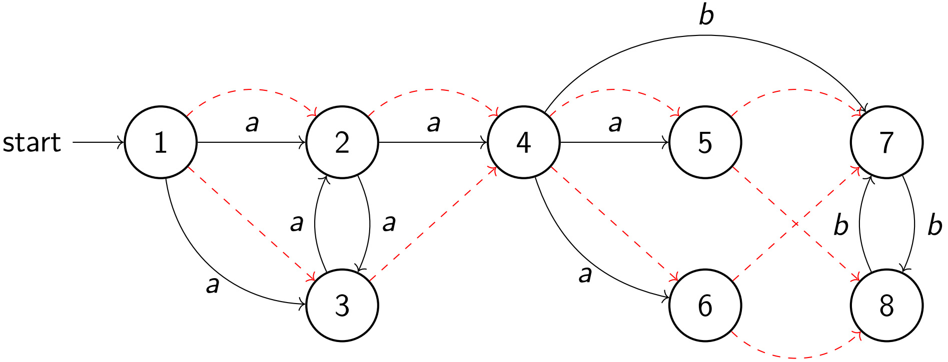
The NFA is not Wheleer, but it is quasi-Wheeler! (P problem)

3. *Alanko et al.: Wheeler languages. Information and Computation. (2021)*

Co-lex orders

Co-lex orders are a generalization of Wheeler orders to **arbitrary NFAs** [4].

Intuition: it's always possible to **partially sort** the states according to the strings reaching them

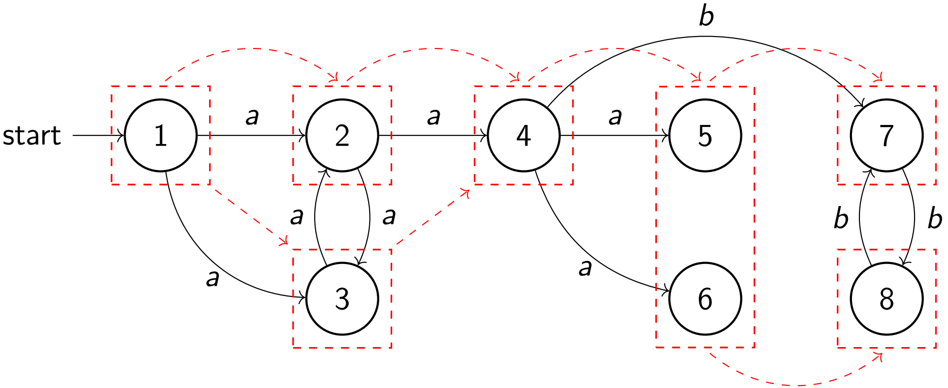


This NFA is neither Wheeler nor quasi-Wheeler. (NP-hard problem)

4. *Cotumaccio N., Prezza N.: On indexing and compressing finite automata. SODA. (2021)*

Maximum co-lex relation

The **maximum co-lex relation** is a partial order on **equivalence classes of states** [5]. For each NFA there exists an unique maximum co-lex relation.

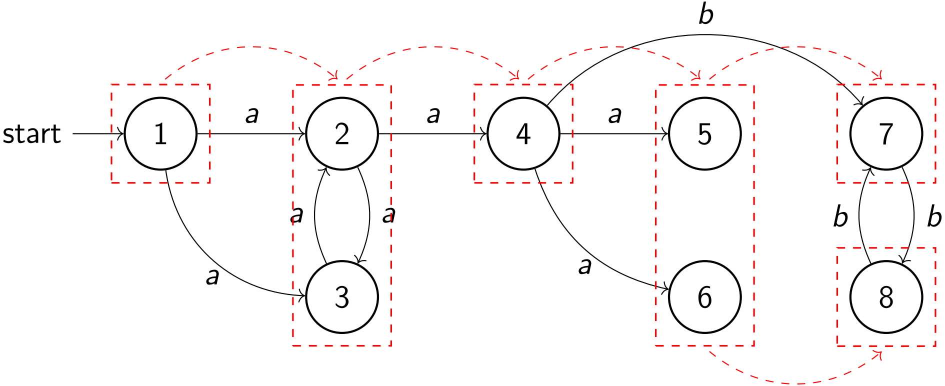


(P problem) States 5 and 6 can be merged into an unique state **without changing the language of the NFA**!

5. *Cotumaccio N.: Graphs can be succinctly indexed for pattern matching in* $O(|E|^2 + |V|^{5/2})$ *time. DCC. (2022)*

CFS order

The **coarsest forward-stable co-lex** (**CFS**) **order** is a partial order on equivalence classes that we introduced in this paper.



Compared to the maximum co-lex order, it manages to **merge more states** together!

Width of orders

- **the space/time efficency** of the index we generate **depends on the width** of the orders we have chosen [4].
- In fact, once we have chosen the order, **we can count the states reached by a pattern** of length **m** in:

$$O(m \cdot w^2 \cdot \log(w \cdot |\Sigma|))$$

4. *Cotumaccio N., Prezza N.: On indexing and compressing finite automata. SODA. (2021)*

Relation among orders

:D	**Existence**	**NP-hard**	**P**
Wheeler order		✓	
Wheeler preorder			✓
CFS order	✓		✓
Max co-lex rel.	✓		✓
Co-lex order*	✓	✓	

decreasing width

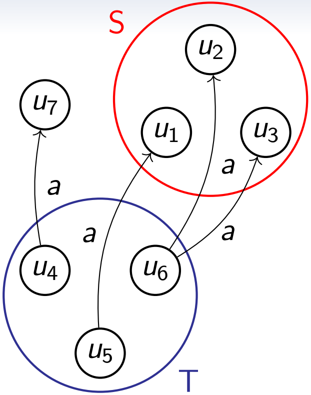
∗ It refers to co-lex orders of minimum width.

Forward-Stability

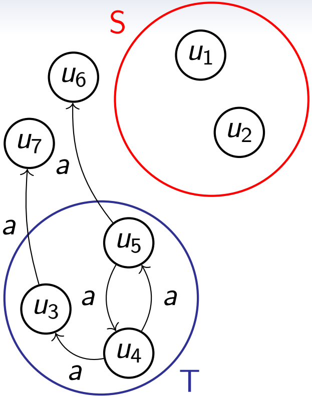
Before presenting our contribution we need to present the notion of **forward-stability**.

1. **Forward-stability between sets of states**.
2. **Forward-stable partitions**.

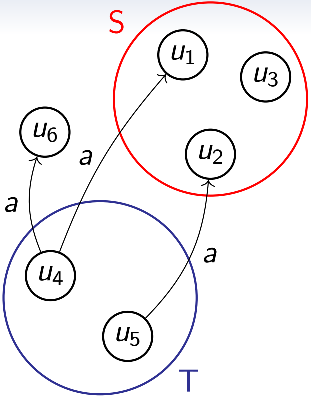
Forward-stability between set of states



All states of S reached by states of T $\Longrightarrow$
S is forward-stable wrt T!



All states of S not reached by states of T $\Longrightarrow$
S is forward-stable wrt T!



some states of S reached by T and some are not $\Longrightarrow$
They're not forward-stable!

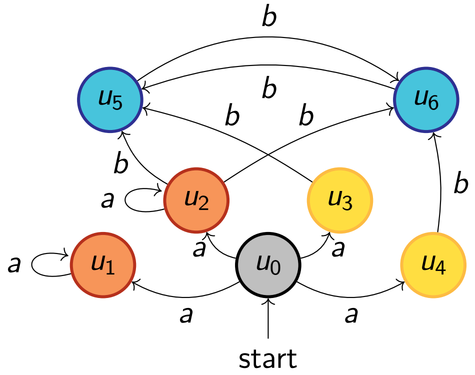
Coarsest forward-stable partition

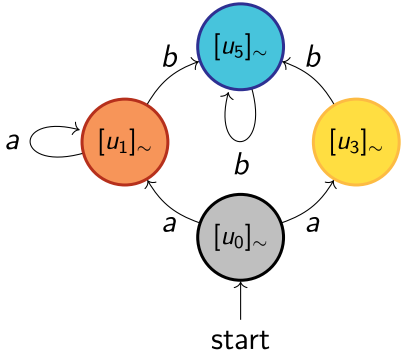
A partition $\mathcal{Q}$ is said to be **forward-stable** if for each parts $S, T \in \mathcal{Q}$ it holds that S **is forward stable wrt** T.

For each NFA there exists a **coarsest forward-stable partition** $Q/_{\sim_{FS}}$, that is, a forward-stable partition formed by the smallest number of parts.

We denote with $\mathcal{A}/_{\sim_{FS}}$ **the quotient automaton defined by the coarsest forward-stable partition**.

Example of a coarsest forward-stable partition





On the left, an automaton $\mathcal{A}$ whose coarsest forward-stable partition is $Q/_{\sim_{FS}} = \{\{u_0\}, \{u_1, u_2\}, \{u_3, u_4\}, \{u_5, u_6\}\}$. **On the right**, the quotient automaton $\mathcal{A}/_{\sim_{FS}}$.

Our contribution

The quotient automata $\mathcal{A}/_{\sim_{FS}}$ and $\mathcal{A}/_{\sim_R}$

$\mathcal{A}/_{\sim_R} = (Q/_{\sim_R}, \delta/_{\sim_R}, \Sigma, s/_{\sim_R})$ $\longrightarrow$ Quotient automaton defined by the **maximum co-lex relation** $\leq_R$ [5].

$\mathcal{A}/_{\sim_{FS}} = (Q/_{\sim_{FS}}, \delta/_{\sim_{FS}}, \Sigma, s/_{\sim_{FS}})$ $\longrightarrow$ Quotient automaton defined by the **coarsest forward-stable partition** [this work!].

Lemma 2

Consider the NFA $\mathcal{A}/_{\sim_R} = (Q/_{\sim_R}, \delta/_{\sim_R}, \Sigma, s/_{\sim_R})$. Then, $Q/_{\sim_R}$ is a **forward-stable partition** of $\mathcal{A}$.

5. *Cotumaccio N.: Graphs can be succinctly indexed for pattern matching in $O(|E|^2 + |V|^{5/2})$ time. DCC. (2022)*

The quotient automata $\mathcal{A}/_{\sim_{FS}}$ and $\mathcal{A}/_{\sim_R}$

$Q/_{\sim_R}$ is a forward-stable partition, however, $Q/_{\sim_{FS}}$ is the **coarsest** forward-stable partition, it follows that:

Theorem 1.3

Consider the partitions $Q/_{\sim_R}$ and $Q/_{\sim_{FS}}$, then the following statement holds:

$$|Q/_{\sim_{FS}}| \leq |Q/_{\sim_R}|$$

Maximum co-lex order of $\mathcal{A}/_{\sim_{FS}}$

A co-lex order of an NFA $\mathcal{A}$ is said to be the **maximum** co-lex order $\leq$ if it is equal to the **union of every co-lex order** of $\mathcal{A}$.

Lemma 3

Consider the NFA $\mathcal{A}/_{\sim_{FS}} = (Q/_{\sim_{FS}}, \delta/_{\sim_{FS}}, \Sigma, s/_{\sim_{FS}})$. Then $\mathcal{A}/_{\sim_{FS}}$ **admits a maximum co-lex order** $\leq$.

The CFS order of an NFA

Definition 10

Consider the NFA $\mathcal{A}/_{\sim_{FS}}$ and its maximum co-lex order $\leq$. Then the **CFS order** $\leq_{FS}$ is the partial order defined as follows:

$$\forall u, v \in Q, \; u \leq_{FS} v \iff [u]_{\sim_{FS}} \leq [v]_{\sim_{FS}}$$

Since the width of these orders is crucial for the time/space efficency of their corresponding indices, we have **compared the widths** of the **maximum co-lex relation** $\leq_R$ and of the **CFS order** $\leq$.

Relation between $\leq_{FS}$ and $\leq_R$

Lemma 6

Let $\leq_R$ and $\leq_{FS}$ be the **maximum co-lex relation** and the **CFS order** of an NFA $\mathcal{A}$, respectively. Then, $\leq_{FS}$ is a **superset** of $\leq_R$.

In other words, this means that;

$$\forall u, v \in Q : u \leq_R v \implies u \leq_{FS} v$$

Relation between $\leq_{FS}$ and $\leq_R$

This directly yields the following result;

Theorem 1.2

Let $\leq_R$ and $\leq_{FS}$ be the **maximum co-lex relation** and the **CFS order** of an NFA $\mathcal{A}$, respectively. Then;

The width of $\leq_{FS}$ *is* **smaller than or equal to** *the width of* $\leq_R$.

Relation between $\leq_{FS}$ and $\leq_R$

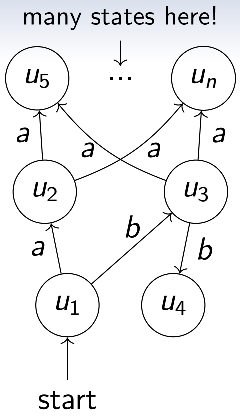
So we demonstrated that the width of $\leq_{FS}$ can be smaller than the width of $\leq_R$, **but how much smaller?**

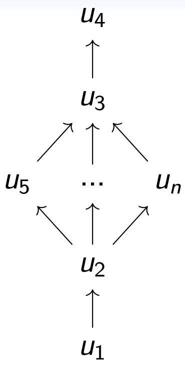
In the next theorem we demonstrated that in some cases it can be **asymptotically smaller**.

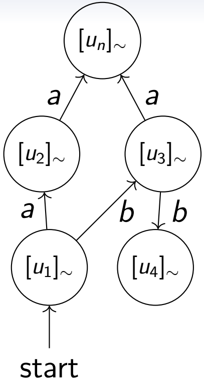
Theorem 2

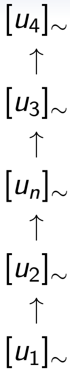
there exist NFAs $\mathcal{A}$ for which $\leq_R$ has **width $\boldsymbol{\Theta}(|\mathbf{Q}|)$** and $\leq_{FS}$ has **width 1**

Proof of Theorem 2









From left to right, **1.** an NFA $\mathcal{A}$ formed by n states, where $\mathcal{A} = \mathcal{A}/_{\sim_R}$. **2.** Hasse diagram of the maximum co-lex relation. **3.** The quotient automaton $\mathcal{A}/_{\sim_{FS}}$, formed by 5 states. **4.** The Hasse diagram of the CFS order.

Time complexity

Corollary 1

The CFS order of an NFA can be computed in $O(|\delta|^2)$ time.

In consideration of these facts, we can conclude that the **CFS order** beats the state-of-the-art competitor, i.e. the **maximum co-lex relation**, in every respect:

- Lower or same width.
- Lower or same number of states.
- Same time complexity.

Thank you for your attention ☺

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