Encoding Co-Lex Orders of Finite State-Automata in Linear Space

36th Annual Symposium on Combinatorial Pattern Matching

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The Burrows-Wheeler transform

The **BWT** is a famous reversible string transformation invented by Burrows and Wheeler [1].

1. **Enhance the compressibility** of strings.
2. Allows the implementation of **indexes for pattern matching**.

$$banana\$ \longrightarrow annb\$aa$$

1. *Burrows M., Wheeler D.: A block-sorting lossless data compression algorithm. SRS Research Report (1994)*

Extending the BWT to NFAs

The BWT has been extended to **nondeterministic finite automata** (NFAs) using **Wheeler orders** [2].

Wheeler order

Total order $\leq$ of nodes **consistent with the strings** reaching the states.

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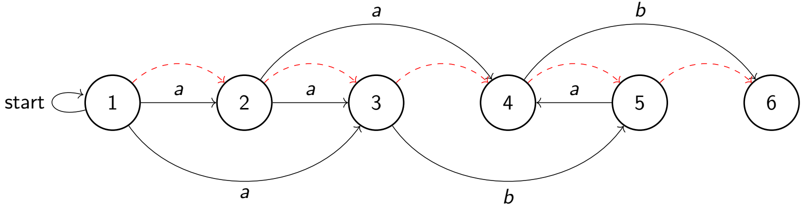

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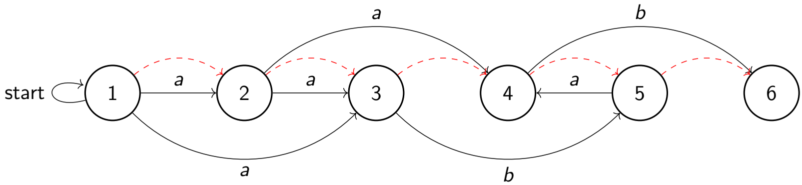

What does "**consistent**" mean?

- **Initial state** is the **first node** of $\leq$ (node u_1)

Strings reaching u_3 and u_4 $\rightarrow$ $I_{u_3} = \{a, aa\}$ and $I_{u_4} = \{aa, aba, aaba\}$

- $u_3 \leq u_4 \implies$ excluding the strings in common $\{aa\}$, strings in I_{u_3} **are smaller than those in** I_{u_4}.

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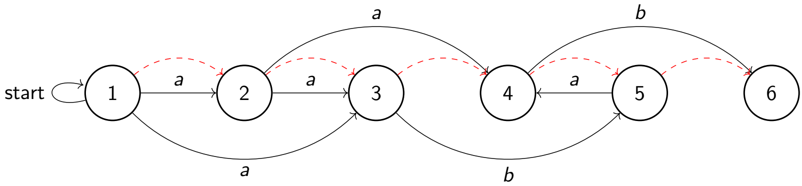

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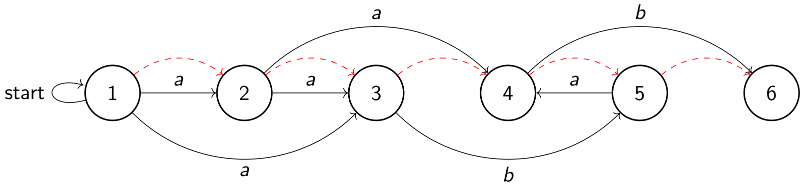

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Wheeler orders: pros and cons

Pros

- NFAs admitting a Wheeler order can be **efficiently compressed and indexed**.

Cons

- Most NFAs do not admit a Wheeler order.
- Recognizing if this is the case is an **NP-complete problem** [3]!

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CFS orders

To address these issues → **CFS orders**! [4]

Coarsest Forward-Stable co-lex (CFS) orders

A CFS order $\leq_{FS}$ is a **partial preorder** on an NFA's states **consistent with the strings reaching them**.

State-Of-The-Art for **indexing** and **compressing** NFAs!

- **Exists** and is **unique** for each NFA.
- **Polynomial time** to compute it.

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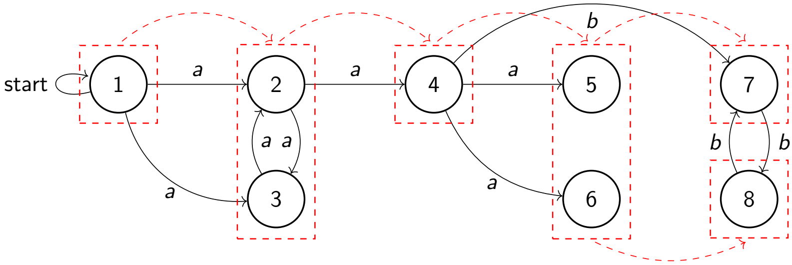
CFS orders

Definition of CFS order $\leq_{FS}$

1) Compute the **quotient NFA** defined by **coarsest forward stable partition**.

2) For every **transitions** $\mathbf{v} \xrightarrow{\mathbf{a}} \mathbf{u}$, $\mathbf{v}' \xrightarrow{\mathbf{a}'} \mathbf{u}'$ in the **quotient NFA**:

- $\mathbf{a} < \mathbf{a}' \implies \mathbf{u} < \mathbf{u}'$
- $(\mathbf{a} = \mathbf{a}') \wedge (\mathbf{v} < \mathbf{v}') \implies \mathbf{u} \leq \mathbf{u}'$



Complexity of CFS Orders

The CFS order can be computed in **polynomial time**, more precisely in $\mathbf{O(m^2)}$ time, with **m** being the number of **transitions in the NFA**.

- Unfeasible for **big data applications** (i.e., to index **pagenome graphs**)

Main open problem: devise a (near) **linear time algorithm** :D

Complexity of CFS Orders for DFAs

For the special case of **DFAs**, a $\mathbf{O(m \log n)}$ time algorithm to compute CFS orders **is already known**. [5]

This algorithm was preceded by a $\mathbf{O(n)}$ **linear space representation** to encode them. [6]

$m \rightarrow$ Number of transitions.

$n \rightarrow$ Number of states.

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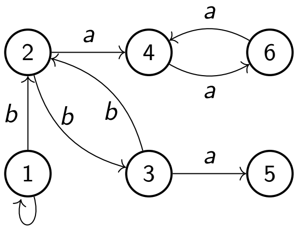
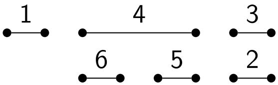
Complexity of CFS Orders for DFAs

Indeed, **in DFAs** the CFS order $\leq_{FS}$ can be **encoded using intervals**



Interval order of $\leq_{FS}$



However, this is **not possible in the case of nondeterminism**

Our main result

Theorem

There exists an **$O(n)$ representation** of $\leq_{FS}$

- Arbitrary partial preorders on V ($|V| = n$) require **$\Omega(n^2)$ bits to be represented** (binary matrix)
- This special class can be encoded in just linear space

Step towards a subquadratic algorithm?

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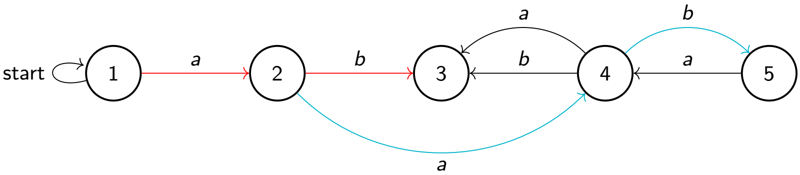
Step towards a subquadratic algorithm?

Characterizing $\leq_{FS}$

Preceding pairs [7]

Let $\mathcal{A}$ be an NFA and Q its set of states.

Consider $(w, z), (u, v) \in Q \times Q$, we say that (w, z) **precedes** (u, v), denoted $(\mathbf{w}, \mathbf{z}) \rightrightarrows (\mathbf{u}, \mathbf{v})$, if there exist α such that $\mathbf{w} \xrightarrow{\alpha} \mathbf{u}$ and $\mathbf{z} \xrightarrow{\alpha} \mathbf{v}$



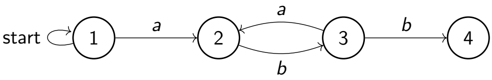
Since $1 \xrightarrow{a} 2 \xrightarrow{b} 3$ and $2 \xrightarrow{a} 4 \xrightarrow{b} 5$, we have $(\mathbf{1}, \mathbf{2}) \rightrightarrows (\mathbf{3}, \mathbf{5})$.

7. *Cotumaccio N.: Graphs can be succinctly indexed for pattern matching in $O(|E|^2 + |V|^{5/2})$ time. DCC. (2022)*

Characterizing $\leq_{FS}$

input-consistency: Incoming transitions are **labeled with same character**.

We denote with $\lambda(\mathbf{u})$ the label of the incoming edges of u.



Ex. $\lambda(3) = b$

Corollary

For every states $u, v \in Q$;

$\mathbf{u} \leq_{\mathbf{FS}} \mathbf{v} \iff$ for each **preceding pair** $(w, z) \rightrightarrows (u, v)$ we have $\lambda(\mathbf{w}) \leq \lambda(\mathbf{z})$.

Ex. $\neg(4 \leq_{FS} 3)$ because $(3, 2) \rightrightarrows (4, 3)$ and $\lambda(3) > \lambda(2)$

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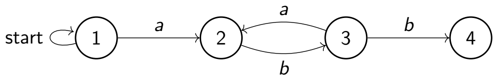

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Overview

Encoding $\leq_{FS}$: $O(n)$ space

- **Linear Extension** $\leq$
- Lexicographically smallest (largest) string **inf** u (**sup** u) and its respective walk **Inf-walk** u (**Sup-walk** u)
- **Inf** (**Sup**) **Conflicts** of u

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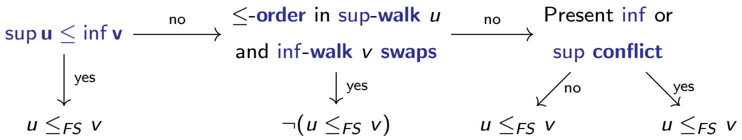
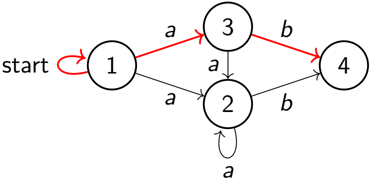
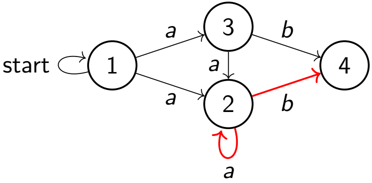


Infima/Suprema

The **Infima** (**Suprema**) are the **smallest** (**largest**) **strings reaching the states** from the initial state. **We can**: compute them in $\mathbf{O(m \log n)}$ [5] stores them in $\mathbf{O(n)}$ [6]





$\inf 4 = ba\#\#\#\#\ldots = ba\#^\omega$

an **infimum walk**: $1 \xrightarrow{\#} 1 \xrightarrow{a} 3 \xrightarrow{b} 4$

$\sup 4 = baaaaa\ldots = ba^\omega$

a **supremum walk**: $2 \xrightarrow{a} 2 \xrightarrow{b} 4$

Lemma

For every two states u, v, $\sup \mathbf{u} \leq \inf \mathbf{v} \implies \mathbf{u} \leq_{\mathbf{FS}} \mathbf{v}$

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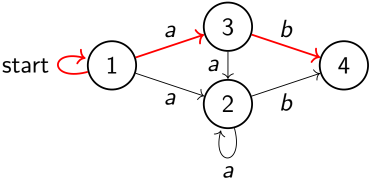

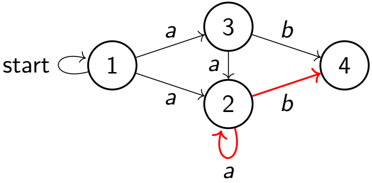

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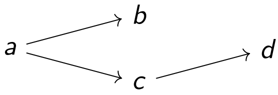
Linear Extension

Second part of our encoding: a linear extension! ($O(n)$ space)

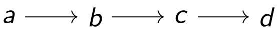
Linear Extension

A total order $\leq$ is a **linear extension** of $\leq_{FS}$ if:

$$\mathbf{u} \leq_{\mathbf{FS}} \mathbf{v} \implies \mathbf{u} \leq \mathbf{v}$$



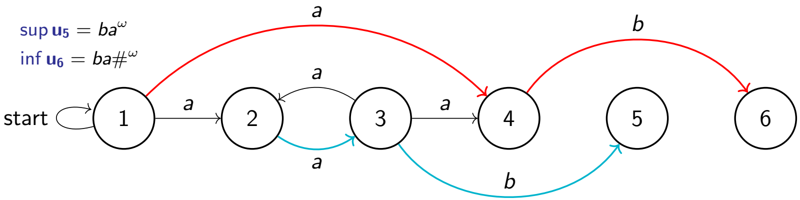
possible **partial order**



possible **linear extension**

The linear extension allows us **to reconstruct** $\leq_{FS}$ when $\sup \mathbf{u} > \inf \mathbf{v}$.

Inf/Sup-Walks Cross



In this case the **linear extension** is $1 \to 2 \to 3 \to 4 \to 5 \to 6$.

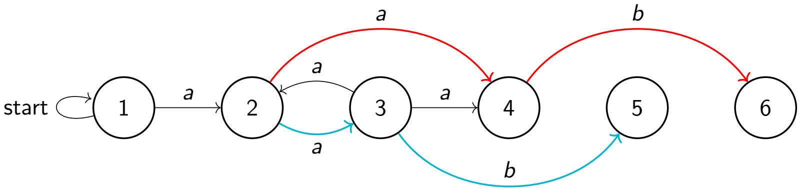
We have sup $\mathbf{5}$ > inf $\mathbf{6}$

Supremum walk to 5: $\mathbf{2 \xrightarrow{a} 3 \xrightarrow{b} 5}$

Infimum walk to 6: $\mathbf{1 \xrightarrow{a} 4 \xrightarrow{b} 6}$

The order swaps!: $2 > 1 \implies 5 \leq_{FS} 6$ does not hold.

Inf/Sup-Walks Meet

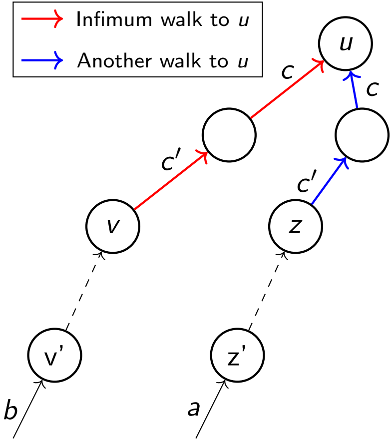


Similar as before, however now the supremum and the infimum **meet at the same node 2**

Special case! We don't know whether $5 \leq_{FS} 6$ holds

We treat this special case separately

Definition Inf/Sup Conflicts



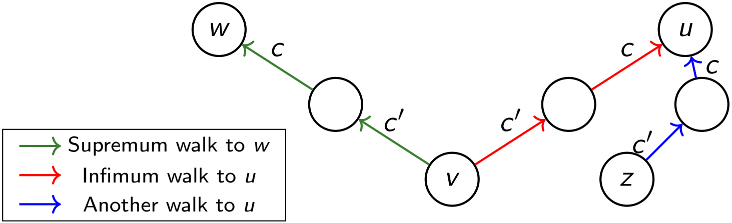
Node **v** is in **Inf conflict** with **u** because:

- **v** is in the **infimum walk** of u
- Node **z** can reach u with the **same labels c′c**
- $v \leq_{FS} z$ **does not hold**

Inf/Sup Conflicts for $\leq_{FS}$

special case seen before: the **preceding pair** of (v, z) is also a **preceding pair** of (w, u) $\implies$ **w $\leq_{FS}$ u does not hold**

w $\leq_{FS}$ u $\iff$ v not in **sup conflict** with **w** or in **inf conflict** with **u**



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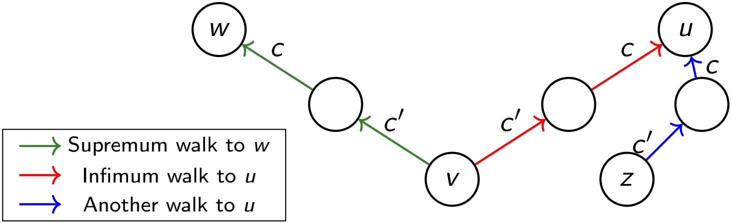

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Representation of $\leq_{FS}$

For every state **u** we save:

- An **infimum walk** to **u**
- A **supremum walk** to **u**
- Its position in the **linear extension**
- Its **inf conflicts**
- Its **sup conflicts**

All of these occupy $O(n)$ total space

Thank you for your attention ☺

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